

# Current profiles and AC losses of a superconducting strip with elliptic cross-section in perpendicular magnetic field

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## Abstract

The case of a hard type II superconductor in the form of strip with elliptic cross-section when placed in transverse magnetic field is studied. We approach the problem in two steps, both based on the critical-state model. First we calculate numerically the penetrated current profiles that ensure complete shielding in the interior, without assuming an *a priori* form for the profiles. In the second step we introduce an analytical approximation that assumes that the current profiles are ellipses. Expressions linking the sample magnetization to the applied field are derived covering the whole range of applied fields. The theoretical predictions are tested by the comparison with experimental data for the imaginary part of AC susceptibility.

## I. INTRODUCTION

The successful use of high- $T_c$  superconductors in the fabrication of conductors for high current applications requires a deep understanding of their response to the magnetic field and, in particular, their ac losses behavior. An important case to study is the magnetic response of a tape made from hard type II superconductor, with a cross-section of elliptic shape, to a magnetic field applied perpendicular to its major axis. This is the configuration very often met in practice and includes the common case of a cylindrical wire in perpendicular field. There were several attempts to study this system, all based in the critical-state model [1]. Within this model framework, the application of a magnetic field in a superconductor results in the penetration of current with a constant density  $J_c$ . The current profile corresponding to a given applied field is distributed in such a way that it keeps the field in the internal region unchanged (so, in the initial magnetization curve, the interior field is kept zero). Once a method is found to obtain the actual shape for the current profiles -also called flux fronts-, all the magnetic properties such as magnetization, ac susceptibility and ac losses can be deduced from it.

A first approach to the problem of a superconducting tape with elliptical cross-section assumed that the flux fronts are ellipses with one constant axis (that in the direction of the applied field) and the other axis varying in order to match the field change [2,3]. However, the assumption of the constant axis was shown to be incorrect in computations of cylindrical wires in transverse field [4], spheres and spheroids [5,6] and also thick strips [7], where it was shown that field profiles detach from the surface.

Another important point is the actual shape of the profiles. Ashkin [4] developed a numerical technique to calculate current penetration profiles by forcing the field on the current penetration profile to vanish. Although the magnetic field in the current-free region was zero, as required by critical-state model, with a precision of only around 10 %, the current profiles results showed a tendency towards a spindle shape (similar to the profiles shown in solid line in Fig. 1). This fact was further pursued by Kuzovlev [8] who demonstrated that,

at least for the case of a sphere, the current profiles have indeed a spindle shape, and not a more smooth shape such as an ellipsoid. However, Bhagwat and Chaddah solved the case of a very thin elliptical tape assuming elliptical flux fronts [9].

Therefore, the situation on such an important system remains unsolved, and questions arise as what the actual shape of current profiles is (elliptical or spindle) and how correct and accurate the various approximations presented up to now are. In this work, we will use a numerical procedure based on the critical-state model to accurately determine the flux fronts that shield the central region of the sample, for any applied field and sample aspect ratio. A key feature of our approach is that, different from the above mentioned models, we will not assume any *a priori* shape for the flux fronts. After briefly describing the main characteristics of the calculated profiles, we will introduce an analytical model which reproduces the features of the actual profiles with enough precision for most needs. We will finally compare our results with experimental data measured on an actual superconducting tape.

## II. NUMERICAL RESULTS

Our numerical model is based on minimizing the magnetic energy of the current distribution after each applied field variation. This approach has been successfully applied to describe the experimental features observed in the initial magnetization [11] as well as the whole magnetization loop and levitation force [12] of superconducting cylinders. The details of the model can be found in [11,12].

Calculated profiles are shown in Fig. 1 (solid lines) for the cases of superconducting tapes of elliptical cross-section with  $b/a=0.1$  (thin ellipse), 1 (circular wire), and 10 (long ellipse), where  $a$  and  $b$  are the ellipse semi-axes. The external field is applied in the direction of the (vertical)  $b$  axis. In order to confirm the validity of our approach, we checked that the field in the non-penetrated region is zero with a precision of around 0.1 %. As a further check of the model, we have found that our calculations for thin elliptical tapes with the

aspect ratio smaller or equal than  $b/a = 0.01$  coincide within numerical accuracy with the analytical formulas for thin ellipses of Bhagwat and Chaddah [9]. This agreement is not observed for thicker samples. Another characteristic of the data is that they follow the known dependence proportional to  $\cos \theta$  predicted for round wires in low applied fields [10]. Finally, it is important to remark that the results from our numerical approach have been further confirmed by calculations based on the analogous, although independent, approach proposed by Brandt [7].

The calculations show several interesting features, which allow us to answer the unsolved questions posed above. First, all the profiles are spindled-shaped, although when the elliptical cross-section of the tape is large in the direction of the applied field (case  $b/a = 10$ , for example) the shape of the profiles resemble more that of an ellipse. Also, in all cases the profiles detach from the surface in the  $b$  axis. The effect is more clear for thick samples than for thin ones. This means that the classical approach of Wilson [2], which assumed no detachment, could work for thin samples but it is not expected to describe accurately the situation of a cylindrical wire in perpendicular applied field. Another interesting feature is that the spacing between the successive field fronts is rather constant for the thicker ellipses but not for the thinner ones.

### III. ANALYTICAL APPROXIMATION

After having obtained the adequate numerical description of the actual current profiles for the superconducting tape with elliptical cross-section, we would like to know if these results could be interpolated by an opportune analytical expression. Here the motivation stems from the fact that analytical approaches have important practical advantages. Therefore we worked out an analytical model that, in spite of its simplicity, reproduces the magnetic results of the numerical model with sufficient accuracy.

The model is based on the assumption that flux fronts are ellipses with semiaxes  $a_0 \varepsilon$  and  $b_0 \varepsilon^{1/n}$ , where  $a_0$  and  $b_0$  are the semiaxes defining the strip's cross-section (see Fig. 2).

Each ellipse is characterised by the independent variable  $\varepsilon$ , with the values ranging from 1 (flux front coinciding with the surface) to 0 (flux front collapsed to the centre of the ellipse). To generalise the previous approaches, an additional parameter characterising the shape of the flux front,  $n$ , was introduced. Indeed, the limit of  $n \rightarrow \infty$  corresponds to the constant axis model used for a round wire [2], while  $n = 2$  would reproduce the assumptions in [9]. We will see that the approximation of flux fronts for tapes with different aspect ratios will require to adjust  $n$  accordingly.

In the critical state approach we utilise here, the current distribution in an elliptic strip with aspect ratio  $\beta = b_0/a_0$  is expressed in polar coordinates as (Fig. 2)

$$j(\varepsilon, r, \theta) = \begin{cases} +j_c & \text{for } r_\varepsilon < r < r_1 \cap -\pi/2 < \theta < \pi/2 \\ 0 & \text{for } r < r_\varepsilon \cup r > r_1 \\ -j_c & \text{for } r_\varepsilon < r < r_1 \cap \pi/2 < \theta < 3\pi/2 \end{cases} \quad (1)$$

where the outer shape of the strip is defined by  $r_1(\theta) = b_0/\sqrt{\beta^2 \cos^2(\theta) + \sin^2(\theta)}$  and the flux front is given as  $r_\varepsilon(\theta) = \varepsilon b_0/\sqrt{\beta^2 \cos^2(\theta) + \varepsilon^{2-2/n} \sin^2(\theta)}$ . The current distribution (1) generates in the ellipse center the magnetic field in the  $y$ -direction

$$\begin{aligned} H_y(\varepsilon) &= \frac{-2j_c}{\pi} \int_0^{\pi/2} \cos\theta (r_1(\theta) - r_\varepsilon(\theta)) d\theta \\ &= -H_p(g(1) - g(\varepsilon)) \end{aligned} \quad (2)$$

where  $H_p = 2J_c b_0/\pi$  is the penetration field for a round wire of radius  $b_0$ , and  $g(x)$  is the auxiliary function

$$g(x) = \frac{x^{1/n} \arctan \sqrt{\frac{\beta^2}{x^{2-2/n}} - 1}}{\sqrt{\frac{\beta^2}{x^{2-2/n}} - 1}}, \quad (3)$$

which depends also on parameters  $\beta$  and  $n$  [13]. An important limit for any  $\beta$  and  $n$  is  $g(0) = 0$ . The magnetic moment per unit length of the strip with the current distribution (1) extended to length  $l \gg a_0, b_0$  in both  $+z$  and  $-z$  directions is calculated as

$$\begin{aligned} \frac{m(\varepsilon)}{l} &= -4j_c \int_0^{a_0} x b_0 \sqrt{1 - \frac{x^2}{a_0^2}} dx \\ -4j_c \int_0^{\varepsilon a_0} x \varepsilon^{1/n} b_0 \sqrt{1 - \frac{x^2}{\varepsilon^2 a_0^2}} dx &= \frac{-4j_c b_0 a_0^3}{3} (1 - \varepsilon^{2+1/n}) \end{aligned} \quad (4)$$

Then, after dividing by the tape cross-section we obtain for the magnetization

$$M(\varepsilon) = \frac{m(\varepsilon)}{\pi a_0 b_0 l} = \frac{-2H_p}{3\beta} (1 - \varepsilon^{2+1/n}) \quad (5)$$

In the case when the whole section of the strip is saturated with the critical current density,  $\varepsilon = 0$  and one can find the penetration field  $H_s = -H_y(0) = H_p g(1)$  and the saturation magnetization  $M_s = |M(0)| = 2H_p/3\beta$ .

Let us now describe the magnetization of the elliptical tape. For a zero-field cooled sample, in the virgin stage of magnetization, the distribution of currents, which is determined by the parameter  $\varepsilon_0$ , should be such that it shields a field  $H_0^{shi}$  equal to the applied field  $H_a$ . From this condition and Eq. (2) the implicit relation linking  $H_a$  with  $\varepsilon_0$  in the initial magnetization curve is found as

$$\begin{aligned} H_i^{shi} &= H_p[g(1) - g(\varepsilon_i)] & H_i^{shi} < H_s \\ \varepsilon_i &= 0 & H_i^{shi} \geq H_s \end{aligned} \quad (6)$$

with  $H_a = H_i^{shi}$  and  $i = 0$ . The initial magnetization  $M_0$ , as a function of  $\varepsilon_0$ , is given by Eq. (5) as

$$M_0(\varepsilon_0) = -\frac{2H_p}{3\beta} \left(1 - \varepsilon_0^{2+\frac{1}{n}}\right) \quad (7)$$

We now analyze the dynamics of flux penetration at the AC field  $H_a = H_m \cos(\omega t)$ , where the maximum field  $H_m$  corresponds to a  $\varepsilon_m$  defined as in Eq. (6) for  $i = m$  and  $H_m^{shi} = H_m$ , so that  $g(\varepsilon_m) = g(1) - H_m/H_p$  if  $H_m < H_s$ , and  $\varepsilon_m = 0$  otherwise.

The amplitude susceptibility, defined as the ratio of the magnetization and the applied field at  $H_a = H_m$  [14] is

$$\chi_a = \frac{M_0(\varepsilon_m)}{H_a(\varepsilon_m)} = -\frac{2}{3\beta} \frac{1 - \varepsilon_m^{2+1/n}}{g(1) - g(\varepsilon_m)} \quad (8)$$

At very small amplitudes, the strip is shielded by currents that flow only in a thin surface shell and  $\varepsilon_m \rightarrow 1$ . The absolute value of the amplitude susceptibility in this limit, denoted  $\chi_0$  [15] is then

$$\chi_0 = \lim_{\varepsilon_m \rightarrow 1} |\chi_a| = \frac{2}{3\beta} \frac{(2n+1)(\beta^2-1)}{(n\beta^2-1)g(1)-n+1} \quad (9)$$

The shielding at low penetrations is a current distribution for which an analytic solution is known [16]. From this solution it follows that the value of  $\chi_0$  is  $\chi_{0,\text{analytic}} = 1 + 1/\beta$ . The correspondence of the value of  $\chi_0$  from (9) with that calculated according to the analytic solution could be then used as a criterion in finding the optimum value of parameter  $n$  in our model for each value of the parameter  $\beta$  as

$$n_{\text{opt}} = \frac{2\beta + 3g(1) - 5}{3\beta^2 g(1) - 4\beta + 1}. \quad (10)$$

When the applied field is descending (from  $H_m$  to  $-H_m$ ) currents in the opposite sense begin to enter from surface to inside the tape. Whereas the already present currents are kept frozen the new entering currents should shield a field  $H_1^{\text{shi}} = (H_m - H_a)/2$  [1,17]. The current profile which shields this field is defined by a  $\varepsilon_1$  similarly as in Eq. (6) for  $i = 1$ . So, the magnetization for a given applied field  $H_a$  (a given  $H_1^{\text{shi}}$ ), considering both new and frozen currents, is

$$\begin{aligned} M_1(\varepsilon_1) &= M_0(\varepsilon_m) - 2M_0(\varepsilon_1) \\ &= \frac{2H_p}{3\beta} (\varepsilon_m^{2+1/n} + 1 - 2\varepsilon_1^{2+1/n}), \end{aligned} \quad (11)$$

where  $\varepsilon_1$  is related with the magnetic field as  $g(\varepsilon_1) = g(1) - (H_m - H_a)/2H_p$  if  $H_a > H^*$ , and  $\varepsilon_1 = 0$  otherwise, and  $H^* = H_m - 2H_s$  is the penetration field of the reverse supercurrents [18]. Notice that if  $H_m < H_s$ , during the reversal stage, reverse currents will never surpass the initial ones,  $\varepsilon_1 \geq \varepsilon_m > 0$ . Analogous magnetization expressions can be easily derived for the ascending part of the cycle (from  $-H_m$  to  $H_m$ ).

We can now proceed with the calculation of  $\chi''$ , the imaginary part of the complex AC susceptibility defined as [19]

$$\chi'' = \frac{2}{\pi H_m^2} \int_{-H_m}^{H_m} M_1(H_a) dH_a \quad (12)$$

This quantity allows to calculate the AC loss [15]. The integral (12) is easily determined in a numerical way, by inserting the corresponding expressions for the magnetization  $M_1(H_a)$  for each stage.

The usefulness of the analytical model described above is that it provides a very convenient approximation to the numerical calculations. In Fig. 1 we show (dashed lines) the current profiles corresponding to the numerical ones. The agreement between both is very good except in the region closest to the  $b$  axis. However, this region is the one that contributes less to the magnetic moment of the sample. Significance of neglecting the difference between the numerical profiles and the analytical approximation is at best evaluated by comparing the results for the magnetization and AC susceptibility. We found for these quantities that the results calculated from the analytical model are hardly distinguishable in practice from the accurate numerical data.

#### IV. EXPERIMENTAL VERIFICATION

Experimental test of these models were performed on a moncore tape from high- $T_c$  superconductor  $\text{Bi}_2\text{Sr}_2\text{Ca}_2\text{Cu}_3\text{O}_{10}$  in silver matrix. The data are presented in Fig. 3, where we show  $\chi''$  as function of the amplitude of the AC field,  $H_m$ . To compare better the shapes of the experimental curves measured at different temperatures with the model proposed here, all the curves were normalized to meet in the maximum point of the  $\chi''$  curve. Our theoretical results correspond to both the analytical and numerical model (they cannot be distinguished in the scale of the picture). It is important to remark that this is a zero-parameter fit, since the value of  $\beta$  is obtained from the actual sample dimensions and  $n$  is chosen from  $\beta$  after Eq. (10). We see clear distinction between the data obtained at superimposed field with respect to those measured without DC field. We explain this by the known fact that applying the DC field much larger than the AC field amplitude limits the actual magnetic fields to a narrow interval on the  $j_c(B)$  dependence, approaching the assumption of field-independent  $j_c$ . Thus, only the curves measured with superimposed DC field should be compared with our model derived under the assumption of constant  $j_c$ . Indeed, we see that the data registered with superimposed DC field coincide nicely with the curve predicted by our model in the region about the maximum. Slight deviations observed



at low fields could be attributed to sample imperfections. On the other side, at large AC amplitudes - up to 0.015 T used in our experiments - the assumption about constant  $j_c$  does not hold anymore because a wide range of local magnetic fields could lead to quite different actual values of  $j_c$ . This, in our opinion, explains also the huge deviation of data measured in zero DC field from our prediction - one can expect under these circumstances a significant deformation of flux front shapes due to the dependence of critical current density on the magnetic field. Similar conclusion was drawn also in another paper tackling the same problem [20]. The narrowing of the  $\chi''(H_a)$  curve due to  $j_c(B)$  was predicted for slabs [19], thin films [21] and cylinders [22].

## V. CONCLUSIONS

We have numerically calculated the current penetration profiles of a strip of elliptical cross-section in a perpendicular applied field, solving some open questions about the shape and properties of the profiles. Moreover, we have presented evidence that, for tapes with any aspect ratio, assuming that the flux fronts are ellipses where the axis in the field direction shrinks as the power  $1/n$  with respect to the shrinking of the perpendicular axis is a good approximation, so they provide good basis for a simple calculation of such properties as magnetization loops, susceptibilities and AC losses.

## VI. ACKNOWLEDGEMENTS

We thank MCyT project BFM2000-0001, CIRIT project 1999SGR00340, and DURSI (Generalitat de Catalunya) as well as the Slovak Grant Agency VEGA for financial support.

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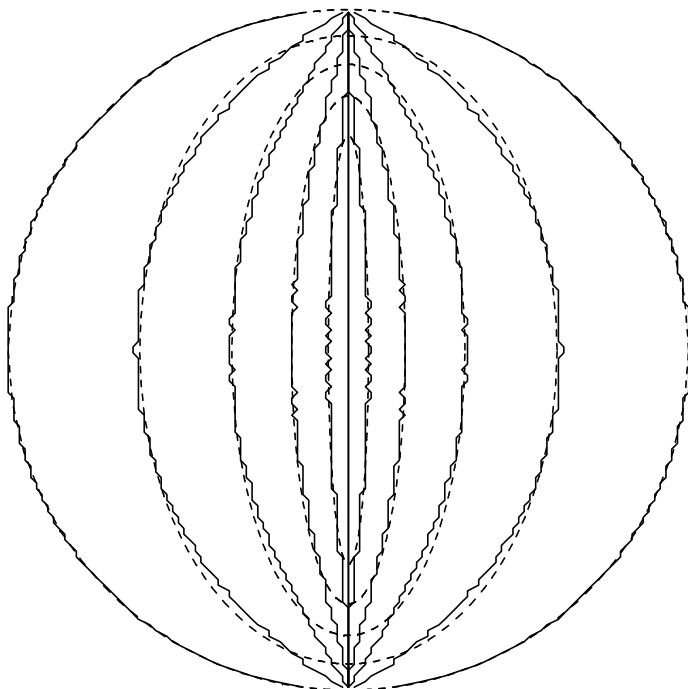
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## FIGURES

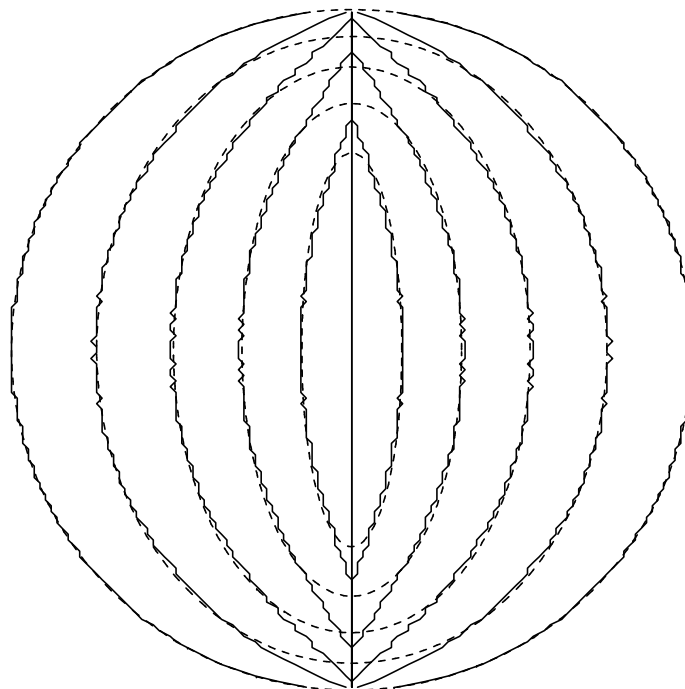
FIG. 1. Current profiles for strips with elliptical cross-section of semiaxes  $b/a = 0.1, 1$ , and  $10$ , corresponding to applied fields  $H_a = 0, 0.2, 0.4, 0.6, 0.8$ , and  $1$ , in units of the penetration field  $H_s$  (from surface inwards). The strip cross-sections have been scaled as circles. Solid lines correspond to the numerical method (Sec. II), while dashed lines to the analytical approximation (Sec. III).

FIG. 2. Sketch of the cross section of the elliptical tape.

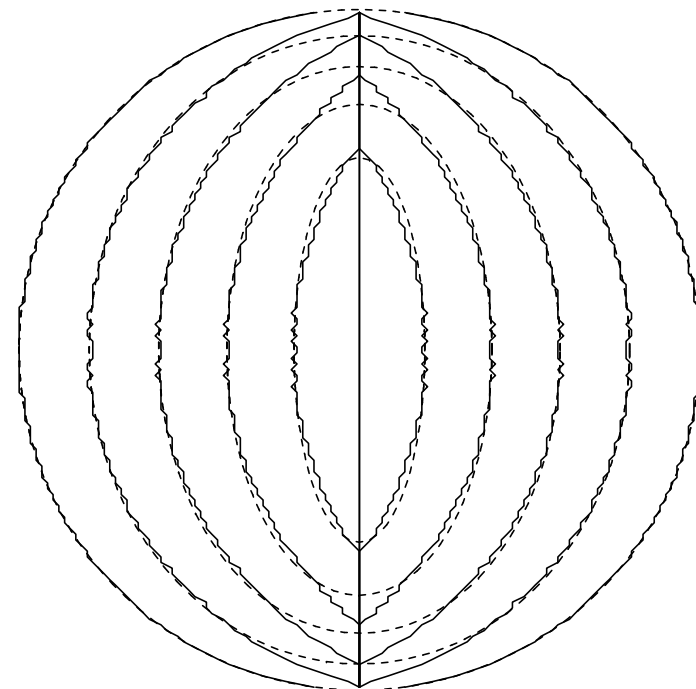
FIG. 3. Imaginary part of the AC susceptibility,  $\chi''$ , as function of the amplitude of the applied ac field,  $H_a$ . The solid line corresponds to theoretical values, and symbols to experimental data obtained from the tape shown in the inset ( $\beta = 0.098$ ), for different temperatures and dc field values.



$b/a=0.1$



$b/a=1$



$b/a=10$

